We conclude this section with a justification of the shell method using step functions. Consider again the solid S in Fig. 9.2.1. We break the region R into thin vertical strips and rotate them into shells, as in Fig. 9.2.8.

What is the volume of such a shell? Suppose for a moment that f has the

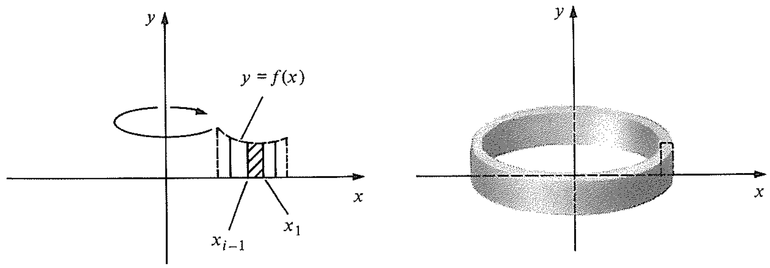

Figure 9.2.8. The volume of a solid of revolution obtained by the shell method.

constant value k_i on the interval (x_{i-1}, x_i). Then the shell is the "difference" of two cylinders of height k_i, one with radius x_i and one with radius x_{i-1}. The volume of the shell is, therefore, $\pi x_i^2 k_i - \pi x_{i-1}^2 k_i = \pi k_i(x_i^2 - x_{i-1}^2)$; we may observe that this last expression is $\int_{x_{i-1}}^{x_i} 2\pi k_i x\,dx$.

If f is a step function on $[a,b]$, with partition $(x_0, \ldots, x_n)$ and $f(x) = k_i$ on (x_{i-1}, x_i), then the volume of the collection of n shells is

$$\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} 2\pi k_i x\,dx;$$

but $k_i = f(x)$ on (x_{i-1}, x_i), so this is

$$\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} 2\pi x f(x)\,dx,$$

which is simply $\int_a^b 2\pi x f(x)\,dx$. We now have the formula

$$\text{volume} = 2\pi \int_a^b x f(x)\,dx,$$

which is valid whenever $f(x)$ is a step function on $[a,b]$. To show that the same formula is valid for general f, we squeeze f between step functions above and below using the same argument we used for the slice method.

Exercises for Section 9.2

In Exercises 1–12, find the volume of the solid obtained by revolving each of the following regions about the y axis and sketch the region.

1. The region under the graph of $\sin x$ on $[0, \pi]$.
2. The region under the graph of $\cos 2x$ on $[0, \pi/4]$.
3. The region under the graph of $2 - (x-1)^2$ on $[0, 2]$.
4. The region under the graph of $\sqrt{4 - 4x^2}$ on $[0, 1]$.
5. The region between the graphs of $\sqrt{3 - x^2}$ and $5 + x$ on $[0, 1]$.
6. The region between the graphs of $\sin x$ and x on $[0, \pi/2]$.
7. The circular region with center $(a, 0)$ and radius r $(0 < r < a)$.
8. The circular region with radius 2 and center $(6, 0)$.
9. The square region with vertices $(4,6)$, $(5,6)$, $(5,7)$, and $(4,7)$.
10. The region in Exercise 9 moved 2 units upward.
11. The region in Exercise 9 rotated by 45° around its center.
12. The triangular region with vertices $(1,1)$, $(2,2)$, and $(3,1)$.
13. The region under the graph of $\sqrt{x}$ on $[0,1]$ is revolved around the y axis. Sketch the resulting solid and find its volume. Relate the result to Example 5 of the previous section.
14. Find the volume in Example 4 by the slice method.
15. A cylindrical hole of radius $\frac{1}{2}$ is drilled through the center of a ball of radius 1. Use the shell method to find the volume of the resulting solid.

Copyright 1985 Springer-Verlag. All rights reserved.